

An exact solution of the Currie-Hill equations in $1 + 1$ dimensional Minkowski space

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Abstract

We present an exact two-particle solution of the Currie-Hill equations of Predictive Relativistic Mechanics in $1 + 1$ dimensional Minkowski space. The instantaneous accelerations are given in terms of elementary functions depending on the relative particle position and velocities. The general solution of the equations of motion is given and by studying the global phase space of this system it is shown that this is a subspace of the full kinematic phase space.

1 Introduction

Relativistic particle mechanics, with instantaneous action-at-a-distance interaction naively contradicts relativistic causality and is thus counter-intuitive. Despite of this apparent difficulty, a consistent theory exists, mainly in the classical domain, but also quantum-mechanically. Nevertheless, relativistic particle physics remains almost completely synonymous with Relativistic Quantum Field Theory. Abandoning the particle alternative is partially due to the famous no-go theorem of Currie, Jordan and Sudarshan [1]. This theorem states that requiring particle positions to satisfy canonical commutation relations excludes the presence of any non-trivial interactions. If we give up this requirement, we can formulate relativistic point mechanics. There are three, essentially equivalent approaches. The first one is called Predictive Relativistic Mechanics (PRM) [2] and is formulated by writing the equations of motion in Newtonian form

$$\ddot{x}_a^i = A_a^i(\{x\}, \{\dot{x}\}), \quad (1.1)$$

where $i = 1, 2, 3$ are space indices, $a = 1, 2, \dots, N$ are particle indices and the accelerations A_a^i occurring in the Newton-equations (1.1) depend on the instantaneous positions x_a^i and velocities \dot{x}_a^i of the particles. Relativistic invariance implies that the accelerations have to satisfy a set of quadratic, partial differential equations, the Currie-Hill (CH) equations [3]:

$$\begin{aligned} \sum_b \left\{ \frac{\partial A_a^i}{\partial v_b^k} + \frac{1}{c^2}(x_a^k - x_b^k)v_b^j \frac{\partial A_a^i}{\partial x_b^j} + \frac{1}{c^2}(x_a^k - x_b^k)A_b^j \frac{\partial A_a^i}{\partial v_b^j} - \frac{1}{c^2}v_b^k v_b^j \frac{\partial A_a^i}{\partial v_b^j} \right\} \\ + \frac{2}{c^2}v_a^k A_a^i + \frac{1}{c^2}A_a^k v_a^i = 0. \end{aligned} \quad (1.2)$$

Here c is the speed of light and we introduced the notation $v_a^i = \dot{x}_a^i$ and Einstein summation convention is used for the (upper) space indices i, j, k but not for particle (lower) indices a, b . The Currie-Hill equations ensure that if we transform the Newton equations to a Lorentz-boosted new coordinate system, the particle trajectories satisfy Newton equations which are instantaneous action-at-a-distance equations in the boosted coordinate system and moreover the equations in the new system are of the same form as (1.1).

One of the difficulties of the relativistic particle dynamics is that unfortunately no explicit solution of the Currie-Hill equations is known, neither in $3+1$ space-time dimensions nor in $1+1$ dimensions. Although the most general 2-particle solution has been found in $1+1$ dimensions [4], but it was given in a very implicit form. There exist approximate solutions in the $1/c^2$ expansion but the absence of explicit exact solutions make the study of further questions like the global structure of the phase

space, symplectic structure, etc. difficult. In this paper we are presenting a completely explicit solution of the Currie-Hill equations in $1+1$ dimensional Minkowski space, written in terms of elementary functions. This explicit solution provides an example in which further questions of the relativistic action-at-a distance approach (conserved quantities, canonical structure, etc.) can be studied transparently.

Having found a solution of the Currie-Hill equations the next natural question is about the existence (and uniqueness) of the 10 integrals corresponding to the Poincaré group. If these exist then one can ask further if a symplectic structure on the phase space (the space of all solutions) can be constructed such that these 10 integrals generate the Poincaré group. An alternative approach to relativistic mechanics [5] can be called canonical. Here a phase space equipped with a symplectic structure is assumed from the beginning, together with the set of 10 generators of the Poincaré group. In this approach consistent relativistic dynamics can be constructed if we can find the particle positions x_a^i , as functions on the phase space and satisfying the Poisson-bracket relations

$$\{P^i, x_a^j\} = \delta^{ij}, \quad \{J^i, x_a^j\} = \epsilon^{ijk} x_a^k, \quad \{K^i, x_a^j\} = \frac{1}{c^2} x_a^i \dot{x}_a^j \quad (1.3)$$

called the world line conditions. Here P^i , J^i , K^i , respectively are the momentum, angular momentum, and Lorentz boost generators, respectively, of the Poincaré group. If we are able to find such particle coordinates, we can calculate the Poisson brackets

$$\{x_a^i, x_b^j\}, \quad (1.4)$$

which must not vanish, otherwise, due to the no-go theorem, there is no interaction. The advantage of the canonical approach is that only the coordinates have to be constructed, the 10 integrals of the Poincaré group are there by construction from the beginning. Provided the set $\{x_a^i\}$, $\{\dot{x}_a^i\}$ are good coordinates on the phase space (at least locally), the accelerations in the Newton-equations (1.1) can be calculated and must satisfy the Currie-Hill equations. There is also a third, essentially equivalent approach [6] which is explicitly covariant. This is not discussed here.

A physical example for relativistic particle interactions is provided by the classical electrodynamics of point charges [7] either in the Feynman-Wheeler formulation or as in Rohrlich's theory. The equations of motion are only known in the post-Coulombian expansion (where the expansion parameter is $1/c^2$). The problem of classical electrodynamics of point charges may be academic, but it is a somewhat simpler analog of the physically relevant problem of motion of compact binaries in general relativity (modeling the bound states of two black holes or two neutron stars). In the latter case the equations of motion are known up to the 3rd post-Newtonian order (up to the terms proportional to c^{-6}) [8] and they satisfy the Currie-Hill equations (in the post-Newtonian perturbative sense). It is not clear

if the expansion can be extended further (the system starts radiating gravitational waves at the 2.5th post-Newtonian order).

Because of the lack of explicit exact solutions it is important to study 1 + 1 dimensional examples, the most famous of which are the exactly solvable Ruijsenaars-Schneider (RS) models [9], the relativistic generalizations of the Calogero-Moser systems. The RS approach is canonical, and the RS systems are not only relativistic, but also integrable for any N . The original motivation of constructing the RS models was their relativistic invariance but later the RS literature was almost entirely concerned with their integrable aspects (there are many applications of the RS models in various areas of physics). Here trajectory variables satisfying the 1 + 1 dimensional version of the world line conditions (1.3) have been constructed but it is not clear if they are good coordinates on the entire phase space and their explicit form in terms of the canonical variables and their commutation relations (1.4) are not known explicitly. There are also further open questions even in the case of RS models (the question of physical non-relativistic limit, for instance) and for this reason it is important to study further examples where all physical questions can be studied more easily. The example we are presenting here is simple enough to do further calculations and to study global questions effortlessly.

We will present our 1 + 1 dimensional solution in the next section. We will construct the conserved quantities associated to the Poincaré generators in section 3. Some conclusion and a list of further questions which can be studied using this example is discussed in the Conclusion section.

2 A solution of the Currie-Hill equations in 1 + 1 dimensions

In this section we will solve the CH equations in 1+1 dimensions for $N = 2$ particles. Rescaling some of the variables we introduce

$$y = x_1 - x_2, \quad u_a = \frac{1}{c}v_a, \quad A_a = c^2\omega_a(y, u_1, u_2), \quad (a = 1, 2). \quad (2.1)$$

In terms of the new variables the CH equations simplify:

$$\begin{aligned} (1 - u_1^2) \frac{\partial \omega_1}{\partial u_1} + (1 - u_2^2 + y\omega_2) \frac{\partial \omega_1}{\partial u_2} - yu_2 \frac{\partial \omega_1}{\partial y} + 3u_1\omega_1 &= 0, \\ (1 - u_1^2 - y\omega_1) \frac{\partial \omega_2}{\partial u_1} + (1 - u_2^2) \frac{\partial \omega_2}{\partial u_2} - yu_1 \frac{\partial \omega_2}{\partial y} + 3u_2\omega_2 &= 0. \end{aligned} \quad (2.2)$$

Not all solutions of the CH equations are physically acceptable in relativistic mechanics. One of the missing ingredients is the relativistic generalization of Newton's third law (action-reaction). In a nonrelativistic two-particle problem we would

require (in addition to the Galilean version of (2.2))

$$m_1 A_1 = -m_2 A_2, \quad (2.3)$$

where m_1 and m_2 are the masses of the particles. As is well known this is equivalent to the statement that the centre of mass of the two-particle system is moving uniformly. In the absence of a proper generalization of the notion of centre of mass for relativistic particles and that of the third law we restrict our attention here to the case of two identical particles. In this case, by symmetry considerations we can assume that $(x_1 + x_2)/2$ moves uniformly and we add to (2.2) the requirement

$$\omega_1 = -\omega_2. \quad (2.4)$$

Even after this simplification the CH equations (2.2) are complicated nonlinear partial differential equations. Hill [4] found the general solution of the equations in an implicit form. Although locally it provides the general solution in terms of two arbitrary functions of two variables, the implicit nature of the solution makes the investigation of global questions difficult. On the other hand, we will see that the particular solution presented in this paper is more suitable for global considerations.

We will look for solutions where the accelerations depend only on the combination

$$\xi = \frac{1 - u_1 u_2}{y} \quad (2.5)$$

of the kinematic variables. It turns out that using the Ansatz

$$\omega_1 = -\omega_2 = f(\xi) \quad (2.6)$$

both equations in (2.2) reduce to the nonlinear ordinary differential equation

$$(f - \xi)f' + 3f = 0. \quad (2.7)$$

To solve (2.7) we first of all write

$$f = \frac{4}{\ell} h^{3/2}, \quad (2.8)$$

where h is a new function characterizing the accelerations and ℓ is our unit of length (it could be scaled out from the problem). In terms of h , (2.7) takes the form

$$\frac{4}{\ell} h^{3/2} h' - \xi h' + 2h = 0. \quad (2.9)$$

We now present the solution of (2.9) assuming that

$$0 < h < 1. \quad (2.10)$$

The solution is given implicitly by

$$h(h-1)^2 = \frac{\ell^2}{4} \xi^2. \quad (2.11)$$

This solution can be made explicit by expressing h in terms of ξ using Cardano's formula. Moreover, the result is an elementary, algebraic function built from square and cubic roots¹. It is, however, easier to study its properties using (2.11) in its original form. Both the variable ξ and the derivative h' can be expressed using (2.11) and its derivative:

$$\xi = \frac{2}{\ell} \sqrt{h}(1-h), \quad h' = \frac{\ell \sqrt{h}}{1-3h}. \quad (2.12)$$

Putting these expressions to (2.9) we see that it is indeed satisfied.

To summarize, we have to find a solution of

$$h(1-h)^2 = Z, \quad Z = \left(\frac{\ell(1-u_1u_2)}{2y} \right)^2 \quad (2.13)$$

in the range $0 < h < 1$ and this parametrizes the acceleration (2.8). The function $Z = h(1-h)^2$ has a single maximum in this range (between 0 and 1) at $h = 1/3$. The maximum value is $4/27$. Thus there is no solution unless

$$0 < Z < \frac{4}{27}, \quad y > \frac{3\sqrt{3}\ell(1-u_1u_2)}{4}. \quad (2.14)$$

This is a new feature of relativistic mechanics. In contrast to Newtonian mechanics, here the initial conditions are not arbitrary. We have to require that the initial conditions satisfy (2.14) in addition to the obvious

$$|u_1| < 1, \quad |u_2| < 1. \quad (2.15)$$

If Z is in the allowed range, there are two solutions for h . We will call the

$$0 < h < \frac{1}{3} \quad (2.16)$$

solution the “good” branch. The mapping between Z and h in the “good” branch is one-to-one if (2.14) is satisfied.

More detailed considerations reveal that (2.14) is only a necessary condition and the phase space should be restricted further. The reason is that it is possible that even if we start from a phase space point satisfying (2.14), during the later (or

¹ $h = \frac{2}{3} + \sqrt[3]{\frac{Z}{2} - \frac{1}{27} + \frac{i}{2}\sqrt{Z(\frac{4}{27} - Z)}} + \sqrt[3]{\frac{Z}{2} - \frac{1}{27} - \frac{i}{2}\sqrt{Z(\frac{4}{27} - Z)}}$

earlier) time evolution of the system we leave this part of the phase space and the accelerations are no longer well defined. We should find a smaller subspace of the full kinematic phase space with the property that starting from here the entire future and past time evolution of the system remains within this subspace. To investigate the global structure of the phase space thus requires the knowledge of the solution of equations of motion, which makes this analysis difficult in general. For the case at hand, the solution of the equations of motion is available and we found that the necessary and sufficient condition for the existence of a uniquely determined and globally well defined (in the above sense) pair of particle trajectories which go through the given phase space point is

$$h_o > 0, \quad y > \frac{\ell(1 - u_1 u_2)}{2\sqrt{h_o}(1 - h_o)}. \quad (2.17)$$

Here h_o depends only on the velocities and is given by

$$p(1 - h_o) = 1 + g + \sqrt{g^2 + \frac{1 + 2g}{9}}, \quad (2.18)$$

where

$$g = \frac{u_1 + u_2}{2 - u_1 - u_2}, \quad p = \frac{2 + u_1 + u_2}{1 - u_1 u_2}. \quad (2.19)$$

It can be shown that

$$h_o \leq \frac{1}{3}. \quad (2.20)$$

The second requirement in (2.17) comes from the “good” branch condition $0 < h < h_o$.

To better understand the meaning of (2.17) and to simplify the formulas we now go to the centre of mass system

$$u_1 = -u_2 = u. \quad (2.21)$$

Here

$$g = 0, \quad p = \frac{2}{1 + u^2} \quad (2.22)$$

and the physical subspace is given by

$$u^2 < \frac{1}{2}, \quad y > \frac{3\sqrt{3}\ell}{4\sqrt{1 - 2u^2}}. \quad (2.23)$$

The general solution of the equations of motion for our system can be given in algebraic form as function of the time t and integration constants. In the centre of mass system² where

$$x_1 = -x_2 = \frac{y}{2} \quad (2.24)$$

²The problem of defining the centre of mass of the system is discussed in section 3. Here we just use the coordinate system defined by (2.21) and (2.24).

the solution becomes very simple and is of the form

$$y = 2b\sqrt{t^2 + B}, \quad u = \frac{bt}{\sqrt{t^2 + B}}. \quad (2.25)$$

There is a single integration constant³ $A > 1$ in this case and the constants b and B are

$$b = \sqrt{\frac{A-1}{A+1}}, \quad B = \frac{\ell^2 A^3}{2(A+1)(A-1)^2}. \quad (2.26)$$

We see that the solution describes a time-reversal symmetric scattering process in which the two particles repel each other. They come in from infinity, gradually approach each other and after the turning point, where the particles stop and reach the minimal relative distance, are receding from each other.

We can also calculate

$$h = \frac{A-1}{2A} \frac{B}{t^2 + B} \quad (2.27)$$

and we see that

$$0 < h < h_{\max}, \quad (2.28)$$

where

$$h_{\max} = \frac{A-1}{2A}. \quad (2.29)$$

The trajectories given by (2.25) exist for any $A > 1$ but the “good” branch condition is only satisfied for

$$1 < A < 3, \quad h_{\max} < \frac{1}{3}. \quad (2.30)$$

Finally we note that in the asymptotic past

$$t \rightarrow -\infty \quad x_1^{(-\infty)} = -x_2^{(-\infty)} \approx -bt \quad (2.31)$$

and similarly

$$t \rightarrow +\infty \quad x_1^{(+\infty)} = -x_2^{(+\infty)} \approx bt, \quad (2.32)$$

which means that the process is not very interesting from the point of view of scattering theory. Although the asymptotic velocities are swapped between the particles, this is a “billiard ball” type scattering, where the time delay vanishes.

³The other, less relevant integration constant is fixed by identifying the origin of the time coordinate with the turning point of the scattering process.

3 Construction of the Poincaré generators

In this section we construct the conserved quantities associated to the three generators of the $1 + 1$ dimensional Poincaré group. We first describe the general strategy of the construction and then carry out the calculation explicitly for our special case.

The three Poincaré generators on the phase space are represented by the differential operators $\hat{\mathcal{H}}$, $\hat{\mathcal{P}}$ and $\hat{\mathcal{K}}$. These are respectively the generators of the time translation, space translation and Lorentz boost. The corresponding functions on the phase space are respectively the Hamiltonian \mathcal{H} , the momentum \mathcal{P} and the (rescaled) centre of mass \mathcal{K} . The generators satisfy the commutation relations of the Poincaré Lie algebra:

$$[\hat{\mathcal{H}}, \hat{\mathcal{P}}] = 0, \quad [\hat{\mathcal{H}}, \hat{\mathcal{K}}] = \hat{\mathcal{P}}, \quad [\hat{\mathcal{P}}, \hat{\mathcal{K}}] = \frac{1}{c^2} \hat{\mathcal{H}}. \quad (3.1)$$

In a canonical mechanical system the Poincaré transformations on phase space functions \mathcal{F} would be generated via the Poisson bracket relations

$$\hat{A}\mathcal{F} = \{A, \mathcal{F}\}, \quad [\hat{A}, \hat{B}] = \widehat{\{A, B\}}. \quad (3.2)$$

The same Poisson bracket relations are the representation of the Poincaré Lie algebra and at the same time the transformation rules of the quantities \mathcal{H} , \mathcal{P} , \mathcal{K} under infinitesimal Poincaré transformations. Here we have no canonical structure but can read off the latter rules by combining (3.1) and (3.2):

$$\begin{aligned} \hat{\mathcal{P}}\mathcal{H} &= \hat{\mathcal{H}}\mathcal{H} = 0, & \hat{\mathcal{K}}\mathcal{H} &= -\mathcal{P}, \\ \hat{\mathcal{P}}\mathcal{P} &= \hat{\mathcal{H}}\mathcal{P} = 0, & \hat{\mathcal{K}}\mathcal{P} &= -\frac{1}{c^2}\mathcal{H}, \\ \hat{\mathcal{K}}\mathcal{K} &= 0, & \hat{\mathcal{H}}\mathcal{K} &= \mathcal{P}, & \hat{\mathcal{P}}\mathcal{K} &= \frac{1}{c^2}\mathcal{H}. \end{aligned} \quad (3.3)$$

From the structure of the above set of transformation rules and the commutation relations (3.1) we can see that a possible strategy of construction is to find a suitable \mathcal{K} satisfying

$$\hat{\mathcal{K}}\mathcal{K} = \hat{\mathcal{P}}\hat{\mathcal{H}}\mathcal{K} = \hat{\mathcal{H}}^2\mathcal{K} = \hat{\mathcal{P}}^2\mathcal{K} = 0. \quad (3.4)$$

All the relations (3.3) are satisfied if we complete the set of conserved quantities by

$$\mathcal{P} = \hat{\mathcal{H}}\mathcal{K}, \quad \mathcal{H} = c^2\hat{\mathcal{P}}\mathcal{K}. \quad (3.5)$$

In a $1 + 1$ dimensional predictive relativistic system the local coordinates on the phase space are the particle positions x_a and velocities v_a and the Poincaré

generators are represented by

$$\begin{aligned}\hat{\mathcal{P}} &= -\sum_a \frac{\partial}{\partial x_a}, \\ \hat{\mathcal{H}} &= \sum_a \left\{ v_a \frac{\partial}{\partial x_a} + A_a \frac{\partial}{\partial v_a} \right\}, \\ \hat{\mathcal{K}} &= \sum_a \left\{ -\frac{x_a v_a}{c^2} \frac{\partial}{\partial x_a} + \left(1 - \frac{v_a^2}{c^2} - \frac{x_a A_a}{c^2} \right) \frac{\partial}{\partial v_a} \right\}.\end{aligned}\tag{3.6}$$

The above representation implements the 1 + 1 dimensional version of the world line conditions (1.3) and in fact the Currie-Hill equations (2.2) are nothing but the requirement that the differential operators (3.6) satisfy the algebra (3.1).

In the case of a single free particle there is no need for the particle label and the differential operators are

$$\hat{\mathcal{P}} = -\frac{\partial}{\partial x}, \quad \hat{\mathcal{H}} = v \frac{\partial}{\partial x}, \quad \hat{\mathcal{K}} = -\frac{xv}{c^2} \frac{\partial}{\partial x} + \left(1 - \frac{v^2}{c^2} \right) \frac{\partial}{\partial v}.\tag{3.7}$$

In this case it is easy to find the general solution of the differential equations (3.4) and in this way we reproduce the familiar formulas

$$\mathcal{K} = -\frac{mx}{\sqrt{1 - \frac{v^2}{c^2}}} + \beta_o, \quad \mathcal{H} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \mathcal{P} = -\frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}},\tag{3.8}$$

where m and β_o are constants. The physical meaning of m is obvious while we can set $\beta_o = 0$ by requiring parity invariance. The physical meaning of the conserved quantities⁴ is here, and in general,

$$\mathcal{H} = E, \quad \mathcal{P} = -P, \quad \mathcal{K} = -\frac{EY}{c^2},\tag{3.9}$$

where E is the total energy, P the total momentum and Y the centre of mass. Note the presence of some minus signs which are due to our conventions.

Next we discuss the case of two symmetric particles where

$$A_1 = -A_2 = f.\tag{3.10}$$

From now on in the rest of this section to simplify the formulas we will set $c = 1$ and use the variables

$$x_1 + x_2 = X, \quad x_1 - x_2 = y, \quad v_1 + v_2 = w, \quad v_1 - v_2 = v.\tag{3.11}$$

⁴Of course the centre of mass is not conserved since its time derivative is given by the total momentum but we will continue to call the set $\{\mathcal{H}, \mathcal{P}, \mathcal{K}\}$ “conserved” quantities.

The generators are

$$\hat{\mathcal{P}} = -2\frac{\partial}{\partial X}, \quad \hat{\mathcal{H}} = w\frac{\partial}{\partial X} + v\frac{\partial}{\partial y} + f\left(\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}\right) \quad (3.12)$$

and

$$\begin{aligned} \hat{\mathcal{K}} = & -\frac{Xw + yv}{2}\frac{\partial}{\partial X} - \frac{Xv}{2}\frac{\partial}{\partial y} - \frac{Xf}{2}\left(\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}\right) - \frac{yw}{2}\frac{\partial}{\partial y} \\ & + (1 - v_1^2)\frac{\partial}{\partial v_1} + (1 - v_2^2)\frac{\partial}{\partial v_2} - \frac{yf}{2}\left(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}\right). \end{aligned} \quad (3.13)$$

We now present the general solution of the differential equations (3.4) in the special case discussed in this paper. In this special case $f = f(\xi)$ and is given by (2.8) and (2.11). In this section we choose our unit of length so that we can set $\ell = 2$. We now define the variables

$$\varepsilon = y(2\xi + f), \quad \Gamma = w^2 + 2(\varepsilon - 2), \quad T = \frac{yv}{\Gamma}, \quad q = \frac{\Gamma}{\varepsilon^2}. \quad (3.14)$$

These satisfy

$$\hat{\mathcal{H}}\varepsilon = 0, \quad \hat{\mathcal{H}}T = 1, \quad \hat{\mathcal{H}}q = \hat{\mathcal{P}}q = \hat{\mathcal{K}}q = 0. \quad (3.15)$$

ε , w and Γ are time-independent and translation invariant, while q is Poincaré invariant. Using these new variables, the general solution is of the form

$$\mathcal{K} = AX + DT + B, \quad \mathcal{H} = -2A, \quad \mathcal{P} = Aw + D, \quad (3.16)$$

where

$$B = B(q), \quad A = \frac{1}{\sqrt{\varepsilon}}g(\varepsilon, q), \quad D = -2w\sqrt{\varepsilon}\frac{\partial g}{\partial \varepsilon} \quad (3.17)$$

and $g(\varepsilon, q)$ has to satisfy the second order (ordinary) differential equation

$$qq = 4(q\varepsilon^2 - 2\varepsilon + 4)\frac{\partial^2 g}{\partial \varepsilon^2} + 4(q\varepsilon - 1)\frac{\partial g}{\partial \varepsilon}. \quad (3.18)$$

The latter has general solution

$$g = g_1(q)\mathcal{R}_+ + g_2(q)\mathcal{R}_-, \quad (3.19)$$

where

$$\mathcal{R}_{\pm} = \sqrt{\frac{1}{q} - \varepsilon \pm \frac{\sqrt{1 - 4q}}{q}}. \quad (3.20)$$

The physical meaning of the conserved quantities can be better understood if we introduce the asymptotic rapidities of the particles. Since as we have seen in the preceding section our system describes the scattering of the two particles, in the asymptotic past we have

$$t \rightarrow -\infty \quad v_1 \rightarrow \tanh \beta_1, \quad v_2 \rightarrow \tanh \beta_2. \quad (3.21)$$

We introduce the combinations

$$2\beta = \beta_1 + \beta_2, \quad 2\theta = \beta_2 - \beta_1. \quad (3.22)$$

(Note that the interaction is repulsive and the phase space can be reduced to $y = x_1 - x_2 > 0$ and $2\theta = \beta_2 - \beta_1 > 0$.) Physical meaning of the conserved quantities can be assessed using the formulas

$$\varepsilon = \frac{4 \cosh 2\theta}{\cosh 2\theta + \cosh 2\beta}, \quad w = \frac{2 \sinh 2\beta}{\cosh 2\theta + \cosh 2\beta}, \quad q = \frac{1}{4} \tanh^2 2\theta. \quad (3.23)$$

The physical meaning of the solution for energy and momentum is given by

$$\mathcal{H} = -2g_1 \frac{\cosh \beta}{\sinh \theta} - 2g_2 \frac{\sinh |\beta|}{\cosh \theta}, \quad \mathcal{P} = 2g_1 \frac{\sinh \beta}{\sinh \theta} + 2g_2 \frac{\text{sign}(\beta) \cosh \beta}{\cosh \theta} \quad (3.24)$$

and shows that the natural choice is

$$g_1(q) = -\frac{m\sqrt{q}}{\sqrt{1-4q}} = -m \sinh \theta \cosh \theta, \quad g_2(q) = 0 \quad (3.25)$$

leading to the usual formulas

$$\mathcal{H} = E = 2m \cosh \theta \cosh \beta, \quad \mathcal{P} = -P = -2m \cosh \theta \sinh \beta \quad (3.26)$$

and

$$\mathcal{K} = B(q) - mX \cosh \theta \cosh \beta + my \sinh \theta \sinh \beta. \quad (3.27)$$

Further it is natural to require that $\mathcal{K} = 0$ in the centre of mass system, where $X = \beta = 0$. This means that we have to choose $B(q) = 0$. This must hold in all coordinate systems since this requirement is Poincaré invariant. Expressing the conserved quantities in terms of the original variables, we finally have

$$\mathcal{H} = 2\mu R, \quad \mathcal{P} = -\frac{\mu w}{R} [1 + \sqrt{1-4q}], \quad \mathcal{K} = -\mu \left[RX + \frac{yvw}{R\varepsilon} \right], \quad (3.28)$$

where

$$\mu = \frac{m}{\sqrt{\varepsilon(1-4q)}}, \quad R = \sqrt{1-q\varepsilon + \sqrt{1-4q}}. \quad (3.29)$$

We note that the centre of mass is given by

$$Y = -\frac{\mathcal{K}}{\mathcal{H}} = \frac{X}{2} + \frac{yvw}{2R^2\varepsilon}, \quad (3.30)$$

which is different from the naive arithmetic mean of the two coordinates. The latter does not define a proper trajectory (not even for two free particles). It is known that the problem of defining the centre of mass of relativistic systems is more complicated than the corresponding Newtonian case. For a discussion of this problem see ref. [10] and references therein. The choice (3.30) is the $1 + 1$ dimensional analog of the Fokker-Pryce centre of inertia and satisfies

$$\hat{\mathcal{H}}Y = V = \frac{P}{E} = \text{const}, \quad \hat{\mathcal{P}}Y = -1, \quad \hat{\mathcal{K}}Y = -YV, \quad (3.31)$$

i.e. the world line conditions for an effective free particle.

4 Conclusion

We have constructed a $1 + 1$ dimensional two-particle relativistic scattering system where the equations of motion can be written in instantaneous action-at-a-distance form and expressed the accelerations as function of the relative distance and particle velocities in terms of elementary functions. We have seen that an interesting new feature with respect to nonrelativistic Newtonian scattering is that initial positions and velocities can not be chosen arbitrarily, the allowed set of initial conditions is a subspace of the full kinematic phase space only.

The reason for studying toy models like the one here is that we can hope to be able to learn something about the unusual features of relativistic point mechanics, which remain valid for more realistic models as well. The following is a list of natural questions that can be studied in any relativistic particle system based on the PRM approach and in particular, can be answered for our simple example.

- Construct the 10 (3 in the case of $1 + 1$ dimensions) conserved quantities of the Poincaré algebra.
- Equip the phase space (defined as the solution space) with symplectic structure such that the above 10 (3) conserved quantities generate the Poincaré transformations on the phase space and in particular the world line conditions (1.3) are satisfied.
- Calculate the Poisson brackets (1.4) and see how the no interaction theorem is circumvented.

- See if a Lagrangian (or action) approach is available for the system.
- (For scattering problems) calculate the time delay (classical analog of scattering phase shifts) as function of asymptotic data (asymptotic momenta of particles).

We have answered the first and last questions in the above list for our simple example and hope to be able to return to the remaining questions in a separate publication.

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